Time Domain Analysis: LTIC Systems

LTIC 🡪 Continuous-time systems

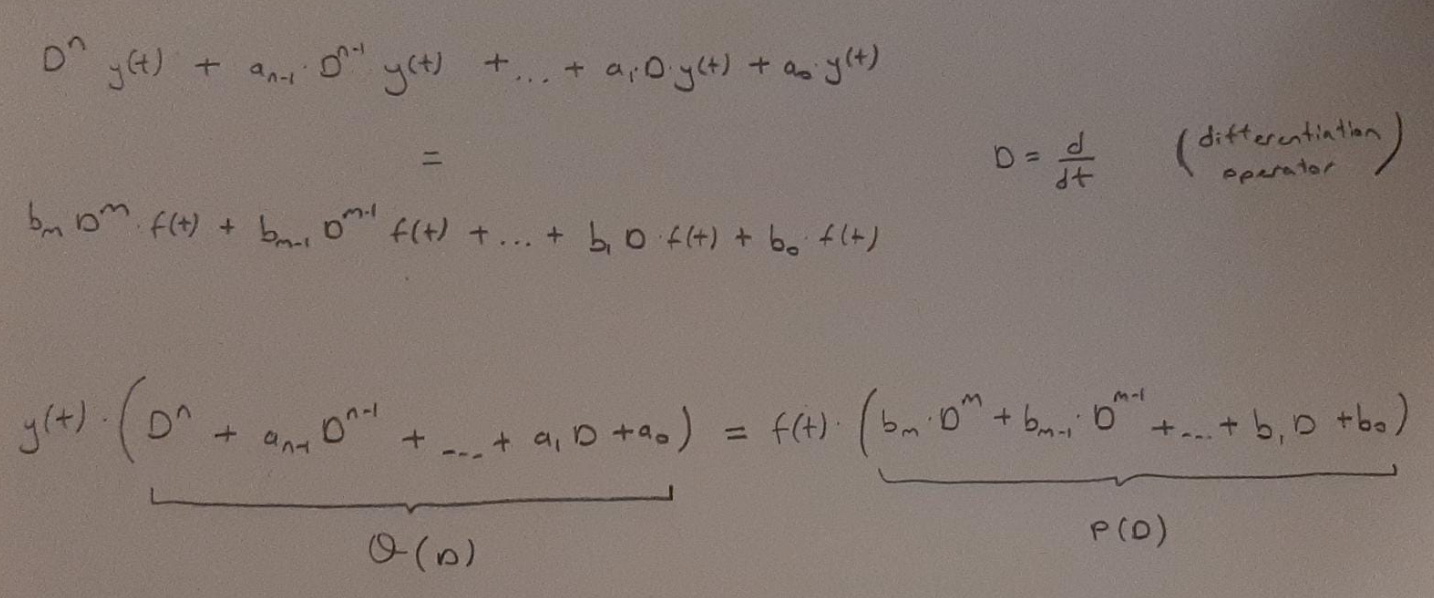
LTID 🡪 Discrete-time systems

LTI: Linear time invariant

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Introduction

Input f(t) and output y(t) of an LTIC system is related by linear differential equations,

Q(D) . y(t) = P(D) . f(t) 🡪 Linear System Equation

Theoratically, m and n can be any value but practically m <= n due to noise considerations.

Response of a System

Total response = Zero-Input Response – Zero-State Response

*Zero-Input Response*

* System response when f(t) = 0
* It is the result of internal system conditions (energy stores) alone
* Independent of external input f(t)

*Zero-State Response*

* System response when all the initial conditions are 0 (zero)
* You just look at the effect of the input signal on the response

The Zero-Input Response

The zero-input response y0(t) is the solution of

Q(D) . y(t) = P(D) . f(t)

when f(t) = 0. That is,

Q(D) . y(t) = P(D) . 0 = 0

Let’s consider the simplest case of 1st order system (when n = 1),

* (D + a0) . y0(t) = 0
* D . y0(t) + a0 . y0(t) = 0
* D . y0(t) = - a0 . y0(t)
* y0(t) = - a0 . y0(t) ------> How can we solve this equation to determine y0(t)?

2 main approaches to solve the equation:

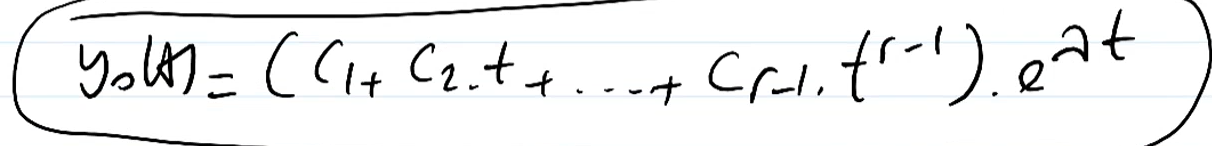
1. Systematic way
2. Heuristic reasoning
   1. Only exponential functions have such property, where the derivative has the same form of itself. Let’s assume that the solution is:
      1. y0(t) = c .
      2. y0(t) = c . . = - a0 . y0(t)
      3. = - a0
      4. Thus the solution is ---> y0(t) = c . (c is arbitrary constant)

To generalize this solution, consider the following equation,

* Q(D) . y0(t) = 0
* When roots are non-repeated:
  + General solution is:
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(Assumes are distinct)

* + where c1, c2, …, cn are arbitrary constants determined by constraints on the solution.
  + ---------------------------------------------------------------------------------------------------------------------
  + Q(λ) = Q(D) |D = λ
    - Q(λ) = (λ – λ1) . (λ – λ2) . … . (λ – λn) --> characteristic polynomial
    - λ1, λ2, …, λn 🡪 characteristic roots
      * parameters or the constants that shows the behaviour of your system
  + (i = 1, 2, …, n) 🡪 characteristic modes
  + ---------------------------------------------------------------------------------------------------------------------
  + There is a characteristic mode for each characteristic root of the system.
  + Therefore, the zero-input response (y0(t)) is a linear combination of the characteristic modes of the system.
* When roots are repeated:
  + (D – λ)2 . y0(t) = 0
  + square means system has 2 roots and they are repeated
  + The solution is:
    - y0(t) = (c1 + c2 . t) .
  + General case:
    - (D – λ)r . y0(t) = 0
  + The characteristic modes are:
    - , t . , t2 . , … , tr-1 .
  + The solution is:
    - 

Example:

* For the given LTIC system,
  + (D2 + 6D + 9) . y(t) = (3D + 5) . f(t)
* Determine y0(t) if y0(0) = 3, y0’(0) = -7 ------> Initial conditions
* --------------------------------------------------------------------------------------------------------------------------------
* First find characteristic polynomial by replacing D with λ
  + Q(λ) = Q(D) |D = λ = (D2 + 6D + 9)|D = λ
  + Q(λ) = λ 2 + 6 λ + 9 ------> CHARACTERISTIC POLYNOMIAL
    - (λ + 3) . (λ + 3)
    - λ1 = -3, λ2 = -3 ---> REPEATED ROOTS
    - Therefore the characteristic modes are and
      * and
* The general solution:
  + y0(t) = (c1 + t . c2) .
  + y0(t) = (c1 + t . c2) .
* To find the unique solution 🡪 determine c1 and c2
  + y0(0) = 3 🡪 c1 = 3
  + y0’(t) = [-3 . (c1 + c2 . t) + c2] . e-3t
  + y0’(t=0) = -3c1 + c2 = -7 🡪 c2 = 2
* Therefore, zero-input response is
  + y0(t) = (3 + 2t) .

Unit Impulse Function

Graphical user interface

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The Unit Impulse Response (h(t))

(t) is used to determine the response of a linear system.

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push this h(t) to output

if f(t) = (t) 🡪 y(t) = h(t)

for an LTIC system,

* Q(D) . y(t) = P(D) . f(t)

h(t) is given by,

* h(t) = bn . (t) + [P(D) . y0(t)] . u(t) ----> u(t) is unit step function
* where bn is the coefficient of nth order in Q(D) and initial conditions are:
  + y0n-1(0) = 1,
  + y0(0) = y0’(0) = y0’’(0) = … = y0n-2(0) = 0

Example:

* Determine h(t) of the following system,
  + (D2 + 3D + 2) . y(t) = D . f(t)
* --------------------------------------------------------------------------------------------------------------------------------
* h(t) = bn . (t) + [P(D) . y0(t)] . u(t) ---> we don’t know bn and y0(t)



* P(D) = D
* n = 2
* We need to find b2 now.
  + b2 . D2 + b1 . D1 = P(D) = D ---> b2 = 0, b1 = 1
  + bm . Dm + bm-1 . Dm-1 + …
* b2 = 0, y0(t) = ?
  + Q(λ) = Q(D) |D = λ = (D2 + 3D + 2)|D = λ
  + Q(λ) = λ 2 + 3λ + 2 = (λ + 1) . (λ + 2)
  + λ1 = -1 λ2 = -2 -----> They are non-repeated
  + y0(t) = c1 . + c2 . + … + cn .
  + We have 2 roots so:
    - y0(t) = c1 . + c2 .
    - y0(t) = c1 . + c2 .
* Now we need to find c1 and c2.
* By definition:
  + y0n-1(0) = 1 y0(0) = 0 n=2
  + y0’(0) = 1
* y0(t) = c1 . + c2 .
* y0(t=0) = c1 + c2 = 0 🡪 c1 = -c2
* y0’(t) = -c1 . e-t – 2 . c2 . e-2t |t=0 = 1
* - c1 - 2c2 = 1
* c1 = 1 and c2 = -1
* y0(t) = -
* h(t) = 0 . (t) + [D . ( - )] . u(t)
* h(t) = ( + 2) . u(t)

The Zero-State Response

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y(t) =

Convolution (integral)

z is actually tau

Some Important Properties of Convolution

The commutative:

* f1(t) \* f2(t) = f2(t) \* f1(t)
* \* is convolution operation

The distributive:

* f1(t) \* [f2(t) + f3(t)] = f1(t) \* f2(t) + f1(t) \* f3(t)

The associative:

* f1(t) \* [f2(t) \* f3(t)] = [f1(t) \* f2(t)] \* f3(t)

The shift:

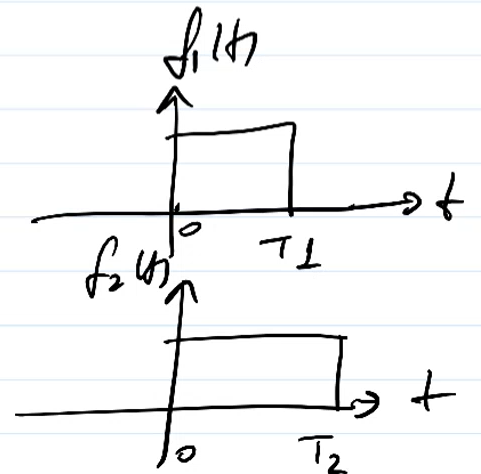
* f1(t) \* f2(t) = c(t)
* f1(t) \* f2(t – T) = c(t – T)
* f1(t – T) \* f2(t) = c(t – T)

Convolution with impulse function:

* f(t) \* (t) = f(t)

The width:

* f1(t) 🡪 T1 duration
* f2(t) 🡪 T2 duration



* f1(t) \* f2(t) = c(t) 🡪 T1 + T2 duration

There are 2 ways to perform convolution:

* By equation
* Visually

Example:

* Consider an LTIC system with h(t) = e-2t . u(t)
* Determine y(t) (zero-state response) for input f(t) = e-t . u(t)
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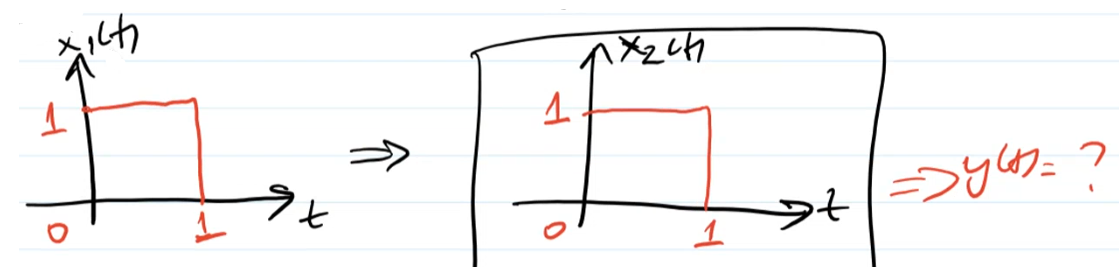
Diagram

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Unit step function

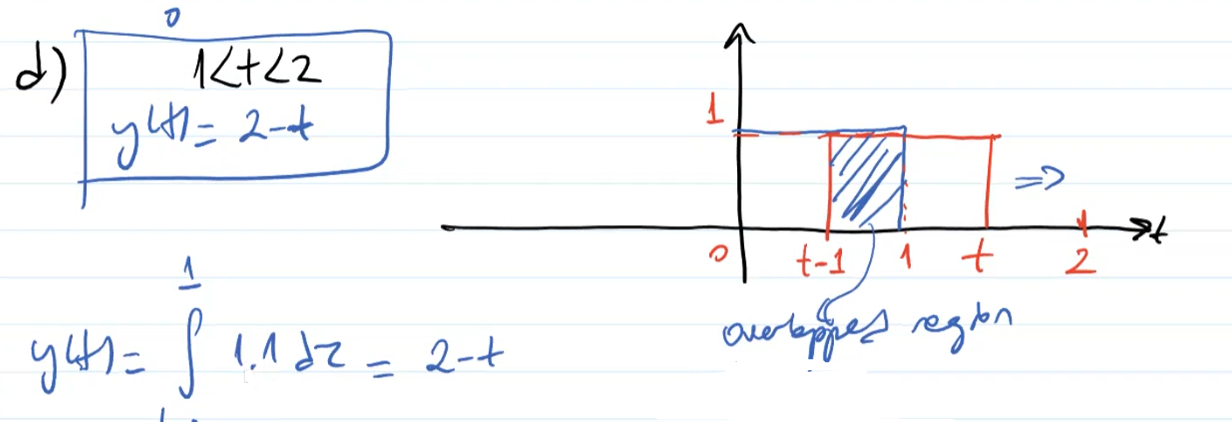
* First way is solve with equation:
* y(t) =
  + f(z) = f(t)|t=z = e-z . u(z)
  + h(t-z) = h(t)|t=t-z = e-2(t-z) . u(z)
  + y(t) = -ze-2(t-z)dz = e-2tzdz = e-2t . ez|0t
  + y(t) = e-2t . (et – 1), t >= 0
  + y(t) = 0, t < 0, then:
  + y(t) = (e-t – e-2t) . u(t) ----> zero state response

Example (Graphical understanding of convolution):

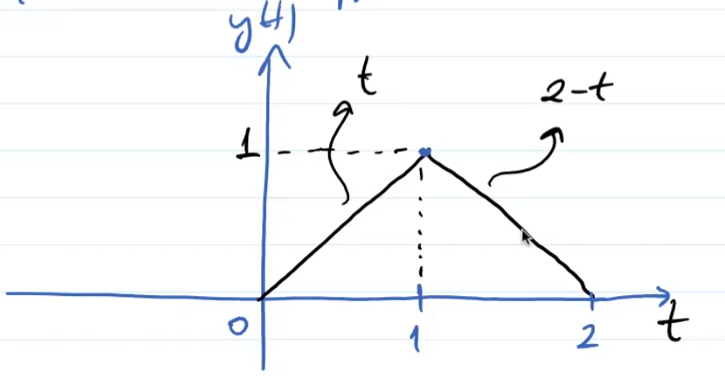
* 
* x1(t) is my input signal
* x2(t) is my system impulse response
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* A picture containing line chart

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* Now we feed x1(t) with system x2(t)
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* blue 🡪 input signal red 🡪 system impulse response
* - infinity to 0 is 0. from t to infinity is also 0.
* Only non-zero region is overlapped region.
* 

**t-1**

* After t >= 2, there is no overlap so we slided over the system. After t >= 2, we obtain the output.
* 

System Stability

If an LTIC system has n distinct characteristic roots 1, 2, …, n, then the zero input response is:

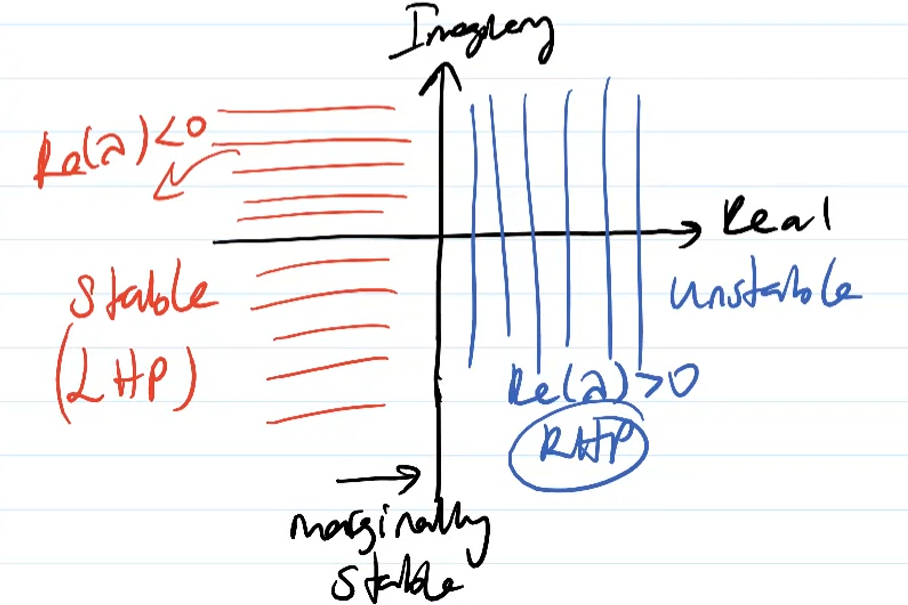
Graphical user interface, application

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0 , Real() < 0 ------> Left hand of complex plane (LHP) – for stable systems

, Real() > 0 ------> Right hand of complex plane (RHP)

=



marginally stable means on top of y axis.

(Asymptotically) Stable: All i are in LHP regardless of repeated and non-repeated.

Unstable: Either one or both of the following conditions;

* at least one root is in the RHP
* there are repeated roots on the imaginary axis

Marginally Stable: There are no roots in the RHP **and** there are some non-repeated roots on the imaginary axis.

Example:

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* --------------------------------------------------------------------------------------------------------------------------------
* The characteristic polynomials are:

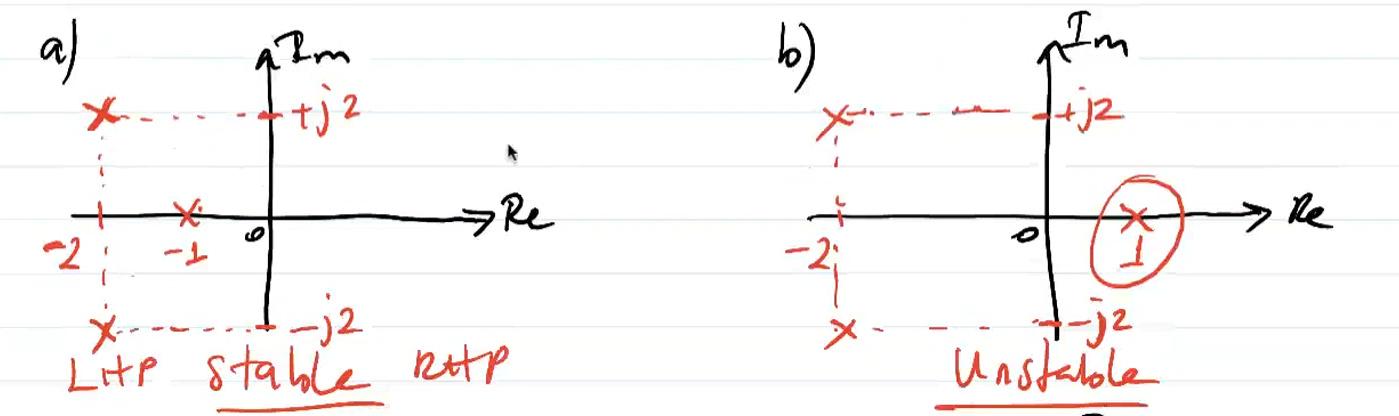
Text, letter

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* Text

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* The characteristic roots are:
* Text, letter

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* Put them in imaginary and real plane:



* 